MANUSCRIPT BOOK 1

OF

SRINIVASA RAMANUJAN

Sanitary Engineer office Check
Accountant General's office
Postal audit office
(Comptroller of Post offices)

$$e^{-2\pi} = \frac{1}{540}\left[1 + \left\{\frac{1}{120}\right.\right.$$

$$\frac{1}{540}\left[1 + \frac{1}{120}\left\{1 + \frac{1}{100}\left(1 + \frac{1}{35}\cdot\overline{1 + \frac{1}{90} + \frac{1}{90\cdot 25}}\right)\right\}\right]$$

$$x \cot x = 1 - B_2 \frac{2x^2}{\lfloor 2} - B_4 \frac{2x^4}{\lfloor 4} - B_6 \frac{2x^6}{\lfloor 6} - \&c$$

(a) $- x \left\{ \cot \frac{x}{2} + \cot \frac{x\omega}{2} + \cot \frac{x\omega^2}{2} \right\}$

$$= 6 \left\{ \frac{B_4}{\lfloor 4} x^4 + \frac{B_{10}}{\lfloor 10} x^{10} + \frac{B_{16}}{\lfloor 16} x^{16} + \&c \right\}$$

$$= x . \frac{\frac{x^6}{\lfloor 6} - \frac{x^{12}}{\lfloor 12} +}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(b) $- \frac{x}{2} \left\{ \cot \frac{x}{2} + \omega^2 \cot \frac{x\omega}{2} + \omega \cot \frac{x\omega^2}{2} \right\}$

$$= 3 \left\{ \frac{B_2}{\lfloor 2} x^2 + \frac{B_8}{\lfloor 8} x^8 + \&c \right\}$$

$$= x . \frac{\frac{x^4}{\lfloor 4} - \frac{x^{10}}{\lfloor 10} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

(c) $- \frac{x}{2} \left\{ \cot \frac{x}{2} + \omega \cot \frac{x\omega}{2} + \omega^2 \cot \frac{x\omega^2}{2} \right\}$

$$= 3 \left\{ B_0 + \frac{B_6}{\lfloor 6} x^6 + \frac{B_{12}}{\lfloor 12} x^{12} + \&c \right\}$$

$$= - x . \frac{\frac{x^2}{\lfloor 2} - \frac{x^8}{\lfloor 8} + \&c}{\frac{x^3}{\lfloor 3} - \frac{x^9}{\lfloor 9} + \&c}$$

$$B_n - \frac{n}{4} . \frac{n-1}{5} \ldots . \frac{n-5}{9} B_{n-6} + \frac{n}{4} . \frac{n-1}{5} \ldots \frac{n-11}{15} B_{n-12}$$

$$+ \&c = \frac{2}{(n+1)(n+2)} (-1)^{\frac{n-2}{6}} \text{ or}$$

$$\frac{2}{(n+1)(n+2)} (-1)^{\frac{n-6}{6}} \text{ or}$$

$$\frac{4}{(n+1)(n+2)} (-1)^{\frac{n-4}{6}}$$

$B_2 = \frac{1}{6}$

$B_8 - \frac{B_2}{3} = -\frac{1}{45}$

$B_{14} - \frac{143}{4} B_8 + \frac{B_2}{5} = \frac{1}{120}$

$B_4 = \frac{1}{30}$

$B_{10} - \frac{5}{2} B_4 = -\frac{1}{132}$

$B_{16} - \frac{286}{3} B_{10} + 4 B_4 = \frac{1}{306}$

$B_0 = -1$

$B_6 - \frac{B_0}{84} = \frac{1}{28}$

$B_{12} - 11 B_6 + \frac{B_0}{455} = -\frac{1}{91}$

$B_{18} - 221 B_{12} + \frac{204}{5} B_6 - \frac{B_0}{1330} = \frac{1}{190}$

$\frac{n+2}{3} B_n = {}^nC_6 \, B_{n-6} B_6 + {}^nC_{12} \, B_{n-12} B_{12}$ &c

when $n-2$ is a multiple of 6.

2-2

i $B_n - \frac{n(n-1)}{2.3} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.3.4.5} B_{n-4} - \&c$

ii $B_n - \frac{n(n-1)}{3.4} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{3.4.5.6} B_{n-4} + \frac{(-1)^{\frac{n}{2}}}{2} = 0$

iii $B_n - \frac{n(n-1)}{4.6} B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{4.6.8.10} B_{n-4} - \&c + \frac{(-1)^{\frac{n}{2}}}{n+1} = 0$

N.B. 0 is excluded in i.

$+ \frac{(-1)^{\frac{n}{2}}}{2^n} = 0$

~~$B_n = B_{n-2}\left\{\frac{n(n-1)}{40} + 33\, n(n-1) \times .000100090\right\}$~~

~~$\log_e B_n = \log_e B_{n-2} + \log n + \log(n-1) - 2\log 2\pi$~~

$$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2} \left(1 - \frac{3}{2^n-1}\right)\left(1 - \frac{8}{3^n-1}\right)\left(1 - \frac{24}{5^n-1}\right)\left(1 - \frac{48}{7^n-1}\right)$$

$$B_n = \frac{n(n-1)}{(n-2)(n-3)} \frac{B_{n-2} \cdot B_{n-2}}{B_{n-4}} \Pi \left\{1 + \frac{(p^2-1)^2}{p^n - 1 - p^4 + \frac{p^4}{p^n}}\right\}$$

$$(n+\tfrac{1}{2})\log_{10} n - 1.23247435\,03\, n + .700120 + \frac{.8362}{n}$$

$$-4\left\{B_2 \frac{x^2}{\lfloor 2} + B_6 \frac{x^6}{\lfloor 6} + B_{10} \frac{x^{10}}{\lfloor 10} + \&c\right\} = x . \frac{\frac{x^3}{\lfloor 3} - \frac{x^7}{2^2 \lfloor 7} + \frac{x^{11}}{2^4 \lfloor 11} - \&c}{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$$

Sin

$\frac{x}{\lfloor 7}$

Sin

$$2\left\{B_0 + B_4 \frac{x^4}{\lfloor 4} + B_8 \frac{x^8}{\lfloor 8} + \&c\right\} = -x . \frac{\frac{x}{\lfloor 1} - \frac{x^5}{2^2 \lfloor 5} + \frac{x^9}{2^4 \lfloor 9} - }{\frac{x^2}{\lfloor 2} - \frac{x^6}{2^2 \lfloor 6} + \frac{x^{10}}{2^4 \lfloor 10} - \&c}$$

3

$\sqrt{2} = 1.4142135623730950488017$

~~$\sqrt{3} = 1.7320$~~

$3 — 64x^{24}$	$7 — \frac{x^{24}}{64}$
$3 — x^3 = \frac{1}{2}$	$7 — x^3 = 1$
$11 — x^3 - x^2 + x = \frac{1}{2}$	$15 —$
$19 — x^3 + x^2 = \frac{1}{2}$	$23 — x^3 + x^2 = 1$
$27 — x^3 + x^2 \sqrt[3]{3} = \frac{1}{2}$	$31 — x^3 + x = 1$
$35 —$	$39 —$
$43 — x^3 + x = \frac{1}{2}$	47
$51 —$	55
$59 —$	63
$67 — x^3 + x^2 + x = \frac{1}{2}$	
$75 —$	

CHAPTER I.

MAGIC SQUARES

Let a be the average, s a row or a column, m middle [illegible] row or column or columns, d a diagonal and w [illegible] the whole sum.

I. When the square contains 3 rows and 3 columns,

i. If s and d are equal write a in the middle and supply the other figures

Sol.— $d_1 + d_2 + m_1 + m_2 = w + 3x$ where x is the reqd figure in middle

$\therefore 4s = 3s + 3x \therefore s = 3x$ or $x = a$.

Cor. The figures in d are in A.P.

Sol.— The sum of the [illegible] and d is s or 3a and 2nd = a $\therefore$ 1st + 3rd = 2a = twice the second [illegible] are in A.P. Similarly in m also.

Ex. 1. Fill up the square when s = 15

6	1	8
7	5	3
2	9	4

2. When s = 27 and all numbers are odd.

15	1	11
5	9	13
7	17	3

ii. When s and d are unequal write $d_1 + d_2 - s$ [illegible] middle [illegible] the numbers [illegible] are in A.P. [illegible]

Ex. 1. [illegible]

Sol. [illegible]

2. Fill up the Square when S = 20, d_1 = 16 and d_2 = 19.

10	2	8
4	5	11
6	13	1

iii. When the diagonals, columns and rows are all different write $\frac{1}{3}(d_1 + d_2 + m_1 + m_2 - w)$ in the middle.

Sol. As in I 1.

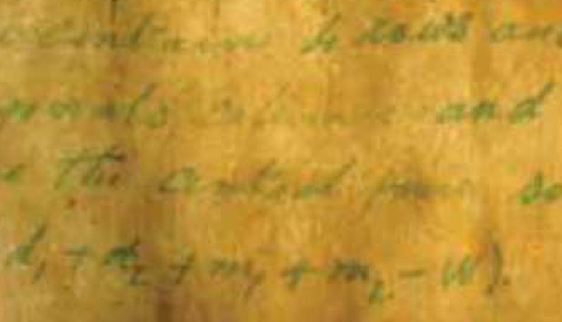
Ex. Fill up the Square when d_1 = 15, d_2 = 19 and the columns and rows are 16, 17, 13, 6, 21, and 18.

4	2	3
8	9	4
7	6	5

2. When an oblong contains 3 rows and 4 columns

A+C = 2B+3D			
A	C+4D	A+4D	C+3D
B+5D	B+3D	B+2D	B
C	A+D	C+2D	A+3D

Ex: Fill up the oblong when a = 8



3. When a square contains 4 rows and 4 columns.

i. When the diagonals, columns and rows are all different, arrange the central four so that the sum may be $\frac{1}{2}(d_1 + d_2 + m_1 + m_2 - w)$.

II. When [illegible]

Fig I

A+P	C+S	D+Q	B+R
D+R	B+Q	A+S	C+P
B+S	D+P	C+R	A+R
C+R	A+R	B+P	D+S

Fig II

A+Q = B+C C+S = D+R

A+P	D+R	D+R	A+S
B+S	C+R	C+R	B+P
C+S	B+R	B+R	C+P
D+P	A+R	A+R	D+S

Ex.

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

1	15	4	[illegible]
12	6	7	9
8	10	11	5
13	3	2	16

4. Where square contains 5 rows and 5 columns.

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+R
B+R	A+T	E+R	D+S	C+P
D+R	C+S	B+P	A+R	E+T

1	58	59	4	5	62	63	8
[illegible]	[illegible]	[illegible]	13	12	51	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	6	37	30	31	40
[illegible]	23	22	45	44	19	18	41
56	[illegible]	14	[illegible]	52	11	[illegible]	49
57	2	3	60	61	6	7	64

Similarly [illegible] squares
Containing [illegible] number [illegible]

CHAPTER II

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots\cdots + \frac{1}{2n}$

$= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots\cdots + \frac{1}{(2n-1)\cdot 2n}$

Sol:– $\frac{1}{(2n-1)\cdot 2n} = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2(2n-1)}$

$\therefore$ R.H.S $= \frac{1}{2}(1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots + \frac{1}{2n-1}) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n})$

$+ \frac{1}{2}(1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots + \frac{1}{2n+1}) + \frac{n}{2n+1} - \frac{1}{2}$

$= (1 + \frac{1}{3} + \frac{1}{5} + \cdots\cdots + \frac{1}{2n-1}) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\cdots + \frac{1}{2n})$

$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots + \frac{1}{2n}) - (1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n})$

$= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots\cdots + \frac{1}{2n}$

Cor. $2\log_e 2 = 1 + \frac{2}{2^3 - 2} + \frac{2}{4^3 - 4} + \frac{2}{6^3 - 6} + \frac{2}{8^3 - 8} + \cdots\cdots$ ad inf.

Sol. R.H.S $= 2(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n})$ when $n = \infty$

Let $n = \frac{1}{dx}$

then the given series $= \frac{2dx}{1+dx} + \frac{2dx}{1+2dx} + \cdots\cdots + \frac{2dx}{1+1}$

$= 2\int_1^2 \frac{1}{x}\,dx = 2\log_e 2$

or thus —

In the Solution of Ex 1. we get $(1 + \frac{1}{3} + \frac{1}{5} + \cdots \frac{1}{2n-1})$ $- (\frac{1}{2} + \frac{1}{4} + \cdots\cdots + \frac{1}{2n})$. When $n = \infty$ this becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c = \log_e 2$ $\therefore$ The reqd. Sum $= 2\log_e 2$

N.B. $\Sigma \frac{1}{n}$ means the sum of the reciprocals of n natural numbers. Therefore $\Sigma \frac{1}{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{2n}$ and $\frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 3} + \cdots\cdots + \frac{1}{2n} = \Sigma \frac{1}{n}/2$ should not be written as $\Sigma \frac{n}{2n}$ which has no meaning according to our convention.

Ex. Show that $\frac{n-1}{n+1} + \frac{n-2}{n+2} + \frac{n-3}{n+3} + \cdots\cdots + \frac{n-n}{n+n}$

$= 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots\cdots + \frac{1}{(2n-1)(2n)(2n+1)}\right\} - \frac{n}{2n+1}$

Sol. We have by II 1.

$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \frac{n}{2n+1} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \cdots \cdots$

$+ \frac{1}{(2n-1)(2n)(2n+1)}$ Multiplying both sides by $2n$

$\frac{2n}{n+1} + \frac{2n}{n+2} + \cdots + \frac{2n}{2n} = \frac{2n^2}{2n+1} + 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \text{\&c} \text{ to } n \text{ terms}\right\}$

$\therefore \left(\frac{2n}{n+1} - 1\right) + \left(\frac{2n}{n+2} - 1\right) + \left(\frac{2n}{n+3} - 1\right) + \cdots + \left(\frac{2n}{2n} - 1\right)$

$= \frac{2n^2}{2n+1} - n + 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \text{\&c to } n \text{ terms}\right\}$

$\therefore \frac{n-1}{n+1} + \frac{n-2}{n+2} + \frac{n-3}{n+3} + \cdots + \frac{n-n}{n+n}$

$= 2n\left\{\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{(2n-1)2n(2n+1)}\right\} - \frac{n}{2n+1}$

2. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} = 1 + \frac{2}{3^3 - 3} + \frac{2}{6^3 - 6} + \cdots + \frac{2}{(3n)^3 - 3n}$

Sol. By proceeding as in II 1 we have R.H.S. $= \frac{1}{3n+1} - \frac{1}{n} = $ L.H.S.

i.e. $1 + \frac{2}{3^3 - 3} + \frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \text{\&c} = \log_e 3.$

3. $\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \cdots + \tan^{-1}\frac{1}{3n+1}$

$= \tan^{-1} 1 + \tan^{-1}\frac{10}{5\cdot 8} + \tan^{-1}\frac{20}{71\cdot 55} + \cdots + \tan^{-1}\frac{10n}{(3n^2+2)(9n$ [illegible]

Cor. $\log_e 3 = \tan^{-1} 1 + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{49} + \tan^{-1}\frac{3}{232} + \tan^{-1}\frac{4}{715} + \text{\&c}$

4. $\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots \frac{1}{4n+1}\right)$

$= 1 + \frac{1}{4^3 - 4} + \frac{1}{8^3 - 8} + \frac{1}{12^3 - 12} + \cdots + \frac{2}{(4n)^3 - 4n}$

$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n+1}\right) + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}\right)$

Sol. By proceeding as in II 1 we have RHS $= \frac{1}{4n+1} - \frac{1}{2} \cdot \frac{1}{2n}$

$- \frac{1}{2} \cdots = \frac{1}{4n+1} - \frac{1}{n} - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$

$= \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{4n+1}\right) - \left(\frac{1}{2n+2} + \frac{1}{2n+4} + \cdots + \frac{1}{4n}\right)$

$= \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n+1}\right)$

Again $\Sigma\frac{1}{4n+1} - \frac{1}{2}\Sigma\frac{1}{n} - \frac{1}{2}\Sigma\frac{1}{n} = \Sigma\frac{1}{4n+1} - 2\Sigma\frac{1}{2n} + \frac{1}{2}\Sigma\frac{1}{2n}$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{4n+1}\right) - 2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{4n}\right)$

$+ \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$

$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{4n+1}$

$+ \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}\right)$

Cor. $1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\&c = \frac{3}{2}\log_e 2$.

5. $\frac{2}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{6n+1}\right)$

$= 1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\frac{2}{18^3-18}+\cdots+\frac{2}{(6n)^3-6n}$

Sol. Proceeding as in III, the sum is $\Sigma\frac{1}{6n+1} - \frac{1}{2}\Sigma\frac{1}{3n}$

$-\frac{1}{3}\Sigma\frac{1}{2n} - \frac{1}{6}\Sigma\frac{1}{n} =$ L.H.S.

Cor. $1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\frac{2}{18^3-18}+\&c = \frac{1}{2}\log_e 3+\frac{1}{3}\log_e 4$

Ex. 1. $\frac{1}{4}\log_e 2 = \frac{1}{2^3-2}+\frac{1}{6^3-6}+\frac{1}{10^3-10}+\frac{1}{14^3-14}+\&c$

2. $\log_e 2 = 1-\frac{2}{2^3-2}+\frac{2}{4^3-4}-\frac{2}{6^3-6}+\&c$

3. $2\left\{1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\frac{2}{12^3-12}+\cdots+\frac{2}{(4n)^3-4n}\right\}$

$= \left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\cdots+\frac{2}{(4n)^3-4n}\right\}+\frac{1}{(4n+1)(4n+2)}$

$+\frac{1}{2}\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+\cdots+\frac{2}{(2n)^3-2n}\right\}$

4. Show that $1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\cdots+\frac{2}{(4n)^3-4n}$

$= \frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{4n+1}\right)$

5. $\frac{1}{3^3-3}+\frac{1}{9^3-9}+\frac{1}{15^3-15}+\&c = \frac{1}{4}\log_e 3-\frac{1}{3}\log_e 2$.

6. $\frac{1}{3}\log_e 2 = 1-\frac{2}{3^3-3}+\frac{2}{6^3-6}-\frac{2}{9^3-9}+\&c$

1. Show that $2\{1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \dots + \frac{2}{(6n)^3-6n}\}$

$+ \frac{1}{3}\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \frac{2}{6^3-6} + \dots + \frac{2}{(2n)^3-2n}\}$

$= 1 + \frac{2}{3^3-3} + \frac{2}{6^3-6} + \dots + \frac{2}{(3n)^3-3n} + \frac{2}{(6n+1)(6n+2)(6n+3)}$

$+ 1 + \frac{2}{2^3-2} + \frac{2}{6^3-6} + \dots + \frac{2}{(6n)^3-6n}$

8. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{7}$

$+ \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{1}{10} + \tan^{-1}\frac{1}{11} + \tan^{-1}\frac{1}{12} + \tan^{-1}\frac{1}{13}$

$= \frac{\pi}{2} + 2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{49} + \tan^{-1}\frac{3}{232} + \tan^{-1}\frac{4}{715}$

9. $2\left(\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \dots \tan^{-1}\frac{1}{2n+1}\right) = \tan^{-1}\frac{n+1}{n}$

$+ \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{4}{137} + \tan^{-1}\frac{6}{667} + \tan^{-1}\frac{8}{2081} +$

$\dots + \tan^{-1}\frac{2n}{8n^4+2n^2+1} +$

$2\left(\tan^{-1}\frac{1}{1.7} + \tan^{-1}\frac{1}{2.19} + \tan^{-1}\frac{1}{3.39} + \dots + \tan^{-1}\frac{1}{n(4n^2+3)}\right)$

10. $\tan^{-1}\frac{1}{n+1} + \tan^{-1}\frac{1}{n+2} + \dots + \tan^{-1}\frac{1}{3n+1} + \tan^{-1}\frac{1}{3n+2}$

$+ \tan^{-1}\frac{1}{3n+5} + \dots + \tan^{-1}\frac{1}{4n+1}$

$= \frac{\pi}{4} + \tan^{-1}\frac{9}{53} + \tan^{-1}\frac{18}{599} + \tan^{-1}\frac{27}{2789} + \dots + \tan^{-1}\frac{9n}{32n^4+2n^2-1}$

$+ \tan^{-1}\frac{4}{137} + \tan^{-1}\frac{8}{2081} + \tan^{-1}\frac{12}{10461} + \dots \tan^{-1}\frac{4n}{128n^4+8n^2+1}$

N.B. $1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \frac{2}{(3x)^3-3x} + \dots + \frac{2}{(nx)^3-nx}$ cannot be expressed as in II 2. for all values of x but 2, 3, 4 and 6 though it can be summed up for all values of x when $n = \infty$. Refer to chapter

6. If $K_n = 3^n(n+\frac{1}{2}) - \frac{1}{2}$. Then

$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{K_n} =$

$$n\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}\right\}$$

$$+(n-1)\left\{\frac{2}{(3K_0+3)^3-(3K_0+3)}+\frac{2}{(3K_0+6)^3-(3K_0+6)}+\cdots+\frac{2}{(3K_1)^3-3K_1}\right\}$$

$$+(n-2)\left\{\frac{2}{(3K_1+3)^3-(3K_1+3)}+\frac{2}{(3K_1+6)^3-(3K_1+6)}+\cdots+\frac{2}{(3K_2)^3-3K_2}\right\}$$

$+ \cdots$ to n terms

Sol. By II 2 we have,

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}$$

$$\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots+\frac{1}{9n+4}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(9n+3)^3-(9n+3)}$$

$$\frac{1}{9n+5}+\frac{1}{9n+6}+\cdots+\frac{1}{27n+13}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(27n+12)^3-(27n+12)}$$

Writing these n times and then adding up all the terms we can get the result.

Cor. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{\frac{3^n-1}{2}}$

$$=n+(n-1)\left(\frac{2}{3^3-3}\right)+(n-2)\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$

$$+(n-3)\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)+\text{\&c to } n \text{ terms}$$

N.B. II 1 & II 2 are very useful in finding the approximate value of $\Sigma\frac{1}{x}$ whether n is small or very great without knowing logarithms, differential and integral calculus. In finding $\Sigma\frac{1}{x}$ it must be remembered that when a_1 and a_n are very great and a_1, a_2, a_3 &c are in A.P. the approximate value of $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$

$$=\frac{2n}{a_1+a_n}$$

$n=3$: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}$

$$=3+\frac{1}{6}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}$$

2. Show that $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1000}$
$= 7\frac{1}{2}$ very nearly.

7. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+3)^2} + \tan^{-1}\frac{2}{(x+5)^2} +$ &c to n terms

$= \tan^{-1}\frac{2n}{x^2+2nx+1}$

Sol. $\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = \tan^{-1}\frac{2}{(x+1)^2}$

$\therefore$ L.H.S $= \left(\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2}\right) + \left(\tan^{-1}\frac{1}{x+2} - \tan^{-1}\frac{1}{x+4}\right)$

$+ \left(\tan^{-1}\frac{1}{x+4} - \tan^{-1}\frac{1}{x+6}\right) + \dots + \left(\tan^{-1}\frac{1}{x+2n-2} - \tan^{-1}\frac{1}{x+2n}\right)$

$= \tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2n} = \tan^{-1}\frac{2n}{x^2+2nx+1}$

Cor. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+3)^2} + \tan^{-1}\frac{2}{(x+5)^2} +$ &c $= \tan^{-1}\frac{1}{x}$.

Sol. Make n infinite in II 7.

Ex. 1. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+2)^2} + \tan^{-1}\frac{2}{(x+3)^2} +$ &c $= \tan^{-1}\frac{2x+1}{x^2+x-1}$

Sol. $\tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+3)^2} +$ &c $= \tan^{-1}\frac{1}{x}$

$\tan^{-1}\frac{2}{(x+2)^2} + \tan^{-1}\frac{2}{(x+4)^2} +$ &c $= \tan^{-1}\frac{1}{x+1}$

$\therefore \tan^{-1}\frac{2}{(x+1)^2} + \tan^{-1}\frac{2}{(x+2)^2} +$ &c $= \tan^{-1}\frac{2x+1}{x^2+x-1}$

N.B. If $x < \frac{\sqrt{5}-1}{2}$ add π to R.H.S.

2. $\tan^{-1}\frac{2}{(x+1)^2} - \tan^{-1}\frac{2}{(x+2)^2} + \tan^{-1}\frac{2}{(x+3)^2} -$ &c $= \tan^{-1}\frac{1}{x^2+x+1}$

3. $\tan^{-1}\frac{1}{2(x+1)^2} + \tan^{-1}\frac{1}{2(x+2)^2} + \tan^{-1}\frac{1}{2(x+3)^2} +$ &c $= \tan^{-1}\frac{1}{2x+1}$

4. $\frac{3\pi}{4} = \tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} +$ &c

5. $\frac{\pi}{4} = \tan^{-1}\frac{1}{2\cdot1^2} + \tan^{-1}\frac{1}{2\cdot2^2} +$ &c $= \tan^{-1}\frac{2}{1^2} - \tan^{-1}\frac{2}{2^2} +$ &c

6. $\frac{\pi}{8} = \tan^{-1}\frac{1}{(1+\sqrt{2})^2} + \tan^{-1}\frac{1}{(1+2\sqrt{2})^2} + \tan^{-1}\frac{1}{(1+3\sqrt{2})^2} +$ &c

8. If $1 = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c$

and $\begin{cases} P_n = A_1 P_{n-1} + A_2 P_{n-2} + A_3 P_{n-3} + \&c \cdots + A_{n-1} P_1 \\ Q_n = A_1 Q_{n-1} + A_2 Q_{n-2} + A_3 Q_{n-3} + \cdots + A_{n-1} Q_1 + A_n Q_0 \\ P_1 = 1 \text{ and } Q_0 = 1 \end{cases}$

then $\frac{P_n}{Q_n}$ approaches x when n becomes greater & greater.

Eg. 1. $x + x^2 = 1$

$x = \frac{0}{1}, \frac{1}{1} \,\Big|\, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$, &c&c

2. $x + x^2 + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2} \,\Big|\, \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}$, &c

3. $x + x^3 = 1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1} \,\Big|\, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{6}, \frac{6}{9}, \frac{9}{13}, \frac{13}{19}$, &c

N.B. If $\frac{p}{q}$ and $\frac{r}{s}$ are two consecutive convergents to x then we may take $\frac{mp + nr}{mq + ns}$ in a suitable manner equivalent to x.

Ex. 1. Find convergents to $\log_e 2$.

Sol. Let $\log_e 2 = x$ then $e^x = 2$

$\therefore 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \&c$

$\therefore x = \frac{0}{1}, \frac{1}{1} \,\Big|\, \frac{1}{1\frac{1}{2}}, \frac{1\frac{1}{2}}{2\frac{1}{6}}, \frac{2\frac{1}{6}}{3\frac{1}{8}}, \frac{3\frac{3}{4}}{5\frac{41}{100}}$ &c

$= \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{9}{13}, \frac{52}{75}, \frac{375}{541}$ &c

2. When $e^{-x} = x$ shew that the convergents to x are

$\frac{1}{2}, \frac{4}{7}, \frac{4}{7}, \frac{148}{261}$, &c

If $\phi(x) = e^x \psi(x)$, then

$$\phi(0) f(0) + \frac{\phi'(0) f'(0)}{\underline{|1}} + \frac{\phi''(0) f''(0)}{\underline{|2}} + \&c$$

$$= \psi(0) f(1) + \frac{\psi'(0) f'(1)}{\underline{|1}} + \frac{\psi''(0) f''(1)}{\underline{|2}} + \&c$$

$$\frac{f(0)}{n} + \frac{f'(0)}{(n+1)\underline{|1}} + \frac{f''(0)}{(n+2)\underline{|2}} + \&c$$

$$= \frac{f(1)}{n} - \frac{f'(1)}{n(n+1)} + \frac{f''(1)}{n(n+1)(n+2)} - \&c$$

$$e^x \left\{ \frac{x}{\underline{|1}} - \frac{2x^2}{2\underline{|2}} + \frac{2^2 x^3}{3\underline{|3}} - \frac{2^3 x^4}{4\underline{|4}} + \&c \right\}$$

$$= \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}}\left(1+\tfrac{1}{3}\right) + \frac{x^4}{\underline{|4}}\left(1+\tfrac{1}{3}\right) + \frac{x^5}{\underline{|5}}\left(1+\tfrac{1}{3}+\tfrac{1}{5}\right)$$

$$+ \frac{x^6}{\underline{|6}}\left(1+\tfrac{1}{3}+\tfrac{1}{5}\right) + \&c$$

$$\frac{1}{n}\left\{\frac{1^n}{\lfloor 1}x + \frac{2^n}{\lfloor 2}x^2 + \frac{3^n}{\lfloor 3}x^3 + \frac{4^n}{\lfloor 4}x^4 + \&c\right\} = e^x f_n(x)$$

1. $\frac{x}{n\lfloor 1} + \frac{x^2}{(n+1)\lfloor 2} + \frac{x^3}{(n+2)\lfloor 3} + \frac{x^4}{(n+3)\lfloor 4} + \frac{x^5}{(n+4)\lfloor 5} + \&c$

$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$

Sol. $L.H.S = \frac{1}{x^{n-1}}\left\{\frac{x^n}{n\lfloor 1} + \frac{x^{n+1}}{(n+1)\lfloor 2} + \frac{x^{n+2}}{(n+2)\lfloor 3} + \&c\right\}$

$= \frac{1}{x^{n-1}}\int x^{n-1}e^x\,dx$

$= \frac{e^x}{x^{n-1}}\left\{\int x^{n-1}dx - \iint x^{n-1}(dx)^2 + \iiint x^{n-1}(dx)^3 - \&c\right\}$

$= \frac{e^x}{x^{n-1}}\left\{\frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+2}}{n(n+1)(n+2)} - \&c\right\} = R.H.S.$

Cor. Theo. —

$\frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+m)} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+m)}$

$= \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \frac{m(m-1)(m-2)}{n(n+1)(n+2)(n+m)}$

&c &c.

$\frac{1}{n+m} = \frac{1}{n} - \frac{m}{n(n+1)} + \frac{m(m-1)}{n(n+1)(n+2)} - \&c$

$\frac{1}{(n+m)\lfloor m} = \frac{1}{n\lfloor m} - \frac{1}{n(n+1)\lfloor m-1} + \frac{1}{n(n+1)(n+2)\lfloor m-2} - \&c$

But $\frac{1}{(n+m)\lfloor m}$ is the coeff. of x^{m+1} in L.H.S and the other is that of x^{m+1} in R.H.S. $\therefore$ L.H.S = R.H.S.

Cor. $\frac{x}{\lfloor 1} + (1+\frac{1}{2})\frac{x^2}{\lfloor 2} + (1+\frac{1}{2}+\frac{1}{3})\frac{x^3}{\lfloor 3} + (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})\frac{x^4}{\lfloor 4} + \&c$

$= e^x\left\{\frac{x}{1\lfloor 1} - \frac{x^2}{2\lfloor 2} + \frac{x^3}{3\lfloor 3} - \frac{x^4}{4\lfloor 4} + \&c\right\}$

Sol. $\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} + \&c = e^x\left\{\frac{x}{n\lfloor 1} - \frac{x^2}{(n+1)\lfloor 2} + \&c\right\}$

Differentiate both Sides with regards to n and then putting 1 for n we can get the result.

$$\ldots - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c$$

$$= \frac{f_0(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \frac{f_3(x)}{n^4} + \&c$$

Sol. By Th 1 we have

$$e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$

$$= \frac{x}{n\,1!} + \frac{x^2}{(n+1)\,2!} + \frac{x^3}{(n+2)\,3!} + \&c$$

changing n to $n+1$ we have

$$e^x\left\{\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c\right\}$$

$$= \frac{1}{n}\left\{\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \&c\right\}$$

$$- \frac{1}{n^2}\left\{\frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 + \&c\right\}$$

$$+ \frac{1}{n^3}\left\{\frac{1^2}{1!}x + \frac{2^2}{2!}x^2 + \frac{3^2}{3!}x^3 + \frac{4^2}{4!}x^4 + \&c\right\}$$

$- \&c = \frac{e^x}{n}f_0(x) - \frac{e^x}{n^2}f_1(x) + \frac{e^x}{n^3}f_2(x) - \&c$ by our supposition. ∴ L.H.S = R.H.S.

N.B. An easier solution for Th 1 is as follows

Let $\phi(n) = \frac{x}{n\,1!} + \frac{x^2}{(n+1)\,2!} + \frac{x^3}{(n+2)\,3!} + \&c$

Then $n\phi(n) = x + \frac{n}{n+1}\frac{x^2}{2!} + \frac{n}{n+2}\frac{x^3}{3!} + \&c$

and $x\phi(n+1) = \frac{x^2}{n+1} + \frac{x^3}{(n+2)\,2!} + \&c$

∴ $n\phi(n) + x\phi(n+1) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \&c = xe^x$

∴ $\phi(n) = e^x\frac{x}{n} - \frac{x}{n}\phi(n+1) = e^x\frac{x}{n} - e^x\frac{x^2}{n(n+1)} + \frac{x^2}{n(n+1)}\phi(n+2)$

&c &c

∴ $\frac{x}{n\,1!} + \frac{x^2}{(n+1)\,2!} + \frac{x^3}{(n+2)\,3!} + \frac{x^4}{(n+3)\,4!} + \&c$

$$= e^x\left\{\frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c\right\}$$